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Cohomology of some nilpotent subalgebras of the Virasoro and Kac-Moody Lie algebras

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Dedicated to I.M. Gelfand on his 75th birthday

Abstract. *The homology of the Lie algebra of algebraic vector fields in the complex line with trivial 3-jet at the point 0 with the coefficients in irreducible highest weight representations of the Virasoro Lie algebra is calculated. The same is done for vector fields with trivial 1-jets at two distinguished points. The class of quasi-finite representations of the Virasoro Lie algebra naturally arises which is the substitute for the class of finite-dimensional representations. The similar results for Kac-Moody Lie algebras are given as well as some conjectures and announcements.*

INTRODUCTION

The Kac-Moody Lie algebras are defined by means of the construction which generalizes directly that of finite-dimensional semisimple Lie algebras. Correspondingly, many results of the representations theory of the Kac-Moody algebras are generalizations of similar results from the finite-dimensional representation theory. In particular, for any Kac-Moody algebra $\mathfrak{g}$ with a specified Cartan decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{f} \oplus \mathfrak{n}_+$ the category of highest weight representations may be defined; it is usually called the category $\mathcal{O}$. The representations from the category $\mathcal{O}$ are locally finite-dimensional with respect to $\mathfrak{n}_-$. In the category $\mathcal{O}$ a semisimple subcategory $\mathcal{K}$ is distinguished; the latter is generated by irreducible representations with (regular) integral dominant highest weight. If $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra then $\mathcal{K}$ is precisely the category of finite-dimensional representations of $\mathfrak{g}$. In general case irreducible representations from $\mathcal{K}$ are similar in many respects to finite-dimensional representations

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of semisimple Lie algebras; in particular, they posess the Bernstein-Gelfand-Gelfand resolution, and the Weyl-Kac character formula is valid for them. They also can be realized in the spaces of sections of invertible sheaves over the flag manifolds.

We study in this paper an analogy for the category $\mathscr{K}$ in the representation theory of the Virasoro Lie algebra Vir. The latter is the central extension of the Lie algebra of vector fields on the circle; it admits a Cartan decomposition $\mathrm{Vir} = \mathscr{L}_1^- \oplus \mathscr{H} \oplus \mathscr{L}_1^+$ (cf. Section 1), and the dimension of the Cartan subalgebra $\mathscr{H}$ is equal to 2. The category $\mathscr{O}$ of representations of Vir, Verma modules and so on can be defined without difficulties. But there is no evident analogy to integral dominant weights. Nevertheless one can indicate in the category $\mathscr{O}$ a class of representations of Vir which bear some resemblance to finite-dimensional ones. To do this we recall some properties of integral dominant highest weight representations of Kac-Moody algebras.

The first property is that such a representation M is always a "top" one. This means that if M is a composition factor of some module N from the category $\mathscr{O}$, then N projects onto M. Moreover, let M be an irreducible module from $\mathscr{K}$ and $\mathscr{K}(M)$ is the minimal subcategory of $\mathscr{O}$ which contains M and such that $\mathscr{O}$ is decomposed into the disjoint sum $\mathscr{K}(M) \amalg \mathscr{O}'$; then there exists a category isomorphism $\mathscr{K}(1) \to \mathscr{K}(M)$, where 1 is a trivial one-dimensional representation.

The second property is that M is the quotient of the Verma module over the submodule generated by l singular vectors where l is the rank of the algebra. And the third property is that the singular carrier of M is 0.

Various arguments suggest one more property. Namely, for any M from $\mathscr{K}$ the space $H_0(\mathfrak{v}; M)$, that is the co-invariant space $M/\ \mathfrak{v}\ M$, is finite-dimensional for any subalgebra $\mathfrak{v}$ of $\mathfrak{g}$ with $\dim(\mathfrak{g}/(\mathfrak{v} + \mathfrak{n}_-)) < \infty$ moreover; this property implies M belonging to $\mathscr{K}$. These assertions are proved only partially. In particular, in Section 4 of this paper we prove that $H_0([\mathfrak{n}_+, \mathfrak{n}_+]; M)$ is finite-dimensional for any M from $\mathscr{K}$; a weakened version of inverse statement is also proved there.

According to this, we call a Vir-module M from the category $\mathscr{O}$ quasi-finite if $H_0([\mathscr{L}_1^+, \mathscr{L}_1^+]; M)$, that is $H_0(\mathscr{L}_3^+; M)$, is finite-dimensional. It turns out that this property of M is equivalent, in the irreducible case, to the union of the analogs of the first and second properties above.

The highest weights of quasi-finite representations of Vir are given by the formulas

$$c = c_{p,q} = 1 - \frac{6(p-q)^2}{pq},$$

$$h = h_{m,n} = \frac{(mq - np)^2 - (q - p)^2}{4pq}$$

where $(p, q) = 1, 0 < m < p, 0 < n < q$. These formulas have occured for the first time in the work [2], where for each $c = c_{p,q}$ a conformal field theory was constructed with $h_{m,n}$ being the dimensions of particles involved.

In this paper we calculate the homology of different subalgebras of Virasoro and Kac-Moody algebras with the coefficients in quasi-finite representations. Section 1 contains an amount of general theory. The homology of the Lie algebra $\mathscr{L}_3 = [\mathscr{L}_1, \mathscr{L}_1]$ with the coefficients in quasi-finite representations of Vir are calculated in Section 2. The Lie algebra $\mathscr{L}_3$ is deformed (inside Vir) into the Lie algebras $\mathscr{L}_{1,1}$ of vector fields in the line which vanish at two given points with the first derivative. In Section 3 we calculate the homology of $\mathscr{L}_{1,1}$ with the coefficients in quasi-finite modules and note that the dimensions of these homology does not change under the deformation $\mathscr{L}_3 \to \mathscr{L}_{1,1}$. The result concerning the 0-dimensional homology of $\mathscr{L}_{1,1}$ may be interpreted as the rule of fusion of particles in the conformal theory. Thus we obtain a proof of these rules. The basic arguments in these two Sections rely on the information on singular vectors in Verma modules obtained in [4].

The last Section 5 contains almost no proofs. Some definitions and results from the forthcoming article of A. Beilinson and B. Feigin [1] are represented here. For a fixed $c = c_{p,q}$ we define a free Abelian group A_c generated by irreducible quasi-finite representations with this c. We introduce an inner product on A_c with respect to which irreducible representations form an orthonormal base. There is also a commutative algebra structure on A_c. The fusion of two particles into one may be interpreted as the multiplication operation in A_c, and the creation of a pair of particles from one as the dual comultiplication operation. The result of Section 3 on the homology of the algebra $\mathscr{L}_{1,1}$ may be stated as follows. Let L be an irreducible quasi-finite module. The comultiplication operation assigns to L a quasi-finite representation $\overline{L}$ of Vir $\oplus$ Vir. The Lie algebra $\mathscr{L}_{1,1}$ is embedded into Vir $\oplus$ Vir, and the spaces $H_*(\mathscr{L}_{1,1}; L)$ and $H^*(\mathscr{L}_{1,1}; \overline{L})$ turn out to be dual to each other. In Section 5 this result is generalized to the case when $\mathscr{L}_{1,1}$ is replaced with the Lie algebra of vector fields in the line which vanish with the first derivative in the points of a given finite set. Some results on co-invariants (that is on 0-dimensional homology) of the Lie algebras of vector fields on non-singular affine curves with the coefficients in quasi-finite modules are also contained in Section 5. In the end of this Section we state some similar results for the Kac-Moody algebra $\hat{\mathfrak{sl}}_2$.

This paper is a product of fusion of two mathematical theories: the representation theory arranged in the category $\mathcal{O}$ and the theory of homology of infinite-dimensional Lie algebras. Both of these theories have originated from the works of I.M. Gelfand, and we express our deep gratitude to him. We are grateful to A. Beilinson for some useful discussions.

1. VERMA MODULES, IRREDUCIBLE MODULES, AND QUASI FINITE MODULES OVER THE VIRASORO ALGEBRA

The following notations and terminology are accepted in this article. The Virasoro algebra Vir is the complex Lie algebra spanned by $e_i\,(i \in \mathbb{Z})$ and z with the commutator relations $[e_i, z] = 0$, $[e_i, e_j] = (j - i)e_{i+j}$ for $i + j \neq 0$, $[e_{-i}, e_i] = 2ie_0 + 1/12\,(i^3 - i)z$. The symbol $\mathscr{L}_i^{\pm}$ $(i \geqslant 0)$ denotes the subalgebra of Vir spanned by $e_{\pm i}, e_{\pm(i+1)}, \ldots$. We write also $\mathscr{L}_i$ instead of $\mathscr{L}_i^+$. By $V_{h,c}$ we denote the Verma module over Vir, that is the module with one generator $v = v_{h,c} \in V_{h,c}$ which generates $V_{h,c}$ freely over $\mathscr{L}_1^+ \subset$ Vir and satisfies the relations $e_0 v = hv$, $zv = cv$, and $\mathscr{L}_1^- v = 0$.

The following is known about the structure of Verma modules $V_{h,c}$ (cf. [5]). For any h, c, h', c' there exists at most one (up to a non-zero multiple) non-trivial Vir-homomorphism $V_{h',c'} \to V_{h,c}$; no such homomorphism exists if $c' \neq c$. The homomorphism $V_{h',c} \to V_{h,c}$ exists in the case when for some positive integers m, n

$$h' = h + mn$$

and h, c satisfy the "Kac equation"

$$\left(h + \frac{1}{24}(m^2 - 1) + \frac{1}{2}(mn - 1)\right)\left(h + \frac{1}{24}(n^2 - 1) + \frac{1}{2}(mn - 1)\right) + \frac{(n^2 - m^2)^2}{16} = 0 \tag{1}$$

All homomorphisms $V_{h',c} \to V_{h,c}$ are the compositions of these homomorphisms.

The curve (1) in the plane $\mathbb{C}^2(h, c)$ is defined as well by the parametric equations

$$h = \frac{1 - m^2}{4}\,t + \frac{1 - mn}{2} + \frac{1 - n^2}{4}\,t^{-1},$$

$$c = 6t + 13 + 6t^{-1}.$$

For h', h, c under these conditions the homomorphism $V_{h',c} \to V_{h,c}$ is denoted by $\varphi_{m,n}(t)$. We norm it by

$$\varphi_{m,n}(t)\,(v_{h',c}) = e_1^{mn}\, v_{h,c} + \ldots ,$$

with "..." belonging to $+_{k \geq 2} e_k V_{h,c}$. Clearly

$$\varphi_{m,n}(t)(v_{h',c}) = \sigma_{m,n}(t)\, v_{h,c},$$

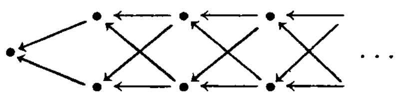
where $\sigma_{m,n}(t) = e_1^{mn} + \ldots \in U(\mathscr{L}_1)$ completely determines $\varphi_{m,n}(t)$. The vector $w = \varphi_{m,n}(t)(v_{h',c}) \in V_{h,c}$ is a "singular vector of weight (h', c)"; this mean that $w \neq 0$, $e_0 w = h'v$, $zw = cw$, and $\mathscr{L}_1^-(w) = 0$.

We see that $\sigma_{m,n}(t)$ is a polynomial in $e_1, e_2, \ldots$ with the coefficients being functions in t (actually polynomials in t, t^{-1}). No explicit formulas for these polynomials are known, but there exist some partial results on them. These will be stated and used below.

For any c all the (proportionality classes of) homomorphisms $V_{h',c} \to V_{h,c}$ compose some disjoint diagrams of the following 5 forms:

(1) $\bullet \longleftarrow \bullet$

(2) $\bullet \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \ldots$

(3) 

(4) and (5) are obtained from (2) and (3) by reversing the arrows. Diagrams of the forms (2), (3) arise only for

$$c = c_{p,q} = 1 - \frac{6(p-q)^2}{pq}$$

where p, q are positive integers prime to each other; in particular, they can arise only for rational $c \leqslant 1$. Diagram of the forms (4), (5) arise only for $c = c'_{p,q} = 26 - c_{p,q}$, in particular only for rational $c \geqslant 25$.

For given p, q diagrams of the form (3) with $c = c_{p,q}$ correspond to the pairs of positive integers m, n with $0 < m < p$, $0 < n < q$, the same diagram corresponds to different pairs (m, n) and (m', n') if and only if $m + m' = p$, $n + n' = q$. The diagram corresponding to the pair (m, n) is shown schematically in Fig. 1. The heavy dots represent Verma modules. Namely, the top dot represents the module $V_{h,c}$ with $c = c_{p,q}$

$$h = h_{m,n} = \frac{(np - mq)^2 - (p-q)^2}{4pq}$$

The other dots represent the modules $V_{h,c}$ with $c = c_{p,q}$, $h = h_{m,n} + N$, where N is the sum of the products of pairs of numbers along the way from

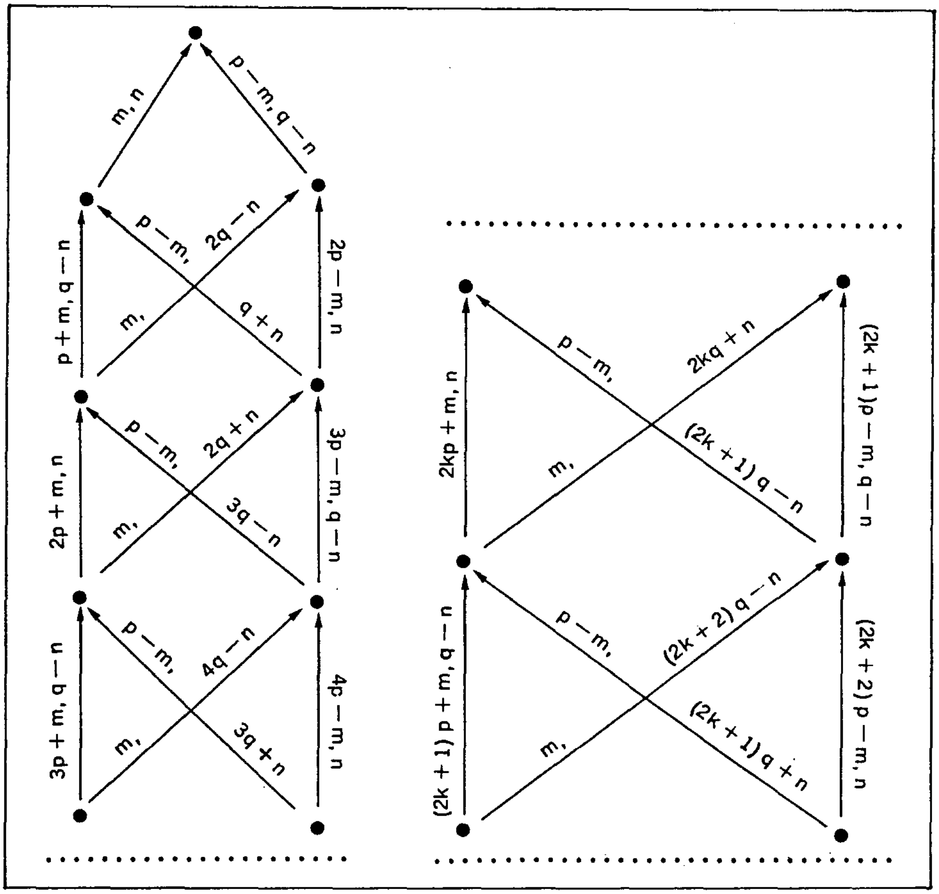


Fig. 1

the dot considered to the top dot. An easy calculation shows that the values of h for the dots of our diagrams are the numbers $h_{m',n'}$, with the following pairs $(m', n') : (p + m, q - n)$ and $(2p - m, n)$ for the two dots next to the top one; $(2p + m, n)$, $(3p - m, q - n)$ for the next two dots, and so on; in other words, these are the pairs indicated in Fig. 1 at the vertical arrows.

The arrows in Fig. 1 represent the homomorphisms $\varphi_{ij}(t)$, where i, j are the numbers written at the arrows, and $t = -q/p$. (For better understanding Fig. 1 it is appropriate to have in mind that $\varphi_{ij}(t) = \varphi_{ji}(t^{-1})$.

Note that the maximal diagrams of form (5) look almost the same: one has only to reverse the arrows (without changing the attached pairs of numbers) and take for t the value $+q/p$ instead of $-q/p$.

Denote by $M_{h,c}$ the maximal submodule of $V_{h,c}$ not including $v_{h,c}$, and set

$L_{h,c} = V_{h,c}/M_{h,c}$. It is known (and evident) that $L_{h,c}$ is an irreducible Vir-module with the highest weight vector, and that any such module is of the form $L_{h,c}$. Moreover, $M_{h,c}$ is the sum of the images of all non-zero homomorphisms $V_{h',c} \to V_{h,c}$ (cf. [5]), and therefore what was said above implies that $M_{h,c}$ is either 0 or is generated (over Vir as well as over $\mathscr{L}_1$) by one or two singular vectors.

DEFINITION. *An irreducible module $L_{h,c}$ is called quasi-finite, if (i) $M_{h,c}$ is generated in $V_{h,c}$ by two singular vectors (is not generated by one singular vector); (ii) $V_{h,c}$ cannot be embedded in another Verma module as a proper submodule.* ∎

In view of the above considerations we have: quasi-finite modules are precisely those of the form $L_{h_{m,n}, c_{p,q}}$ with $c_{p,q}$ and $h_{m,n}$ as above, $(p, q) = 1$, $0 < m < p$, $0 < n < p$. With the exception of the module $L_{0,0} = \mathbb{C}$ (occuring when $p = 3$, $q = 2$, $m = n = 1$) all the modules $L_{h,c}$ are infinite-dimensional. The term "quasi-finite" is motivated by the following conjecture.

CONJECTURE 1.1. *The following two properties of the module $L_{h,c}$ are equivalent.*

(i) *$L_{h,c}$ is quasi-finite.*

(ii) *For any subalgebra $\mathfrak{v}$ of Vir with $\dim(Vir/(\mathfrak{v} + \mathscr{L}_1^-)) < \infty$ the co-invariant space $L_{h,c}/\mathfrak{v} L_{h,c}$ is finite-dimensional.* ∎

The implication (ii) ⇒ (i) will be proved in Section 2. Moreover, we shall prove that the quasi-finiteness of $L_{h,c}$ is equivalent to the finite-dimensionality of $L_{h,c}/\mathscr{L}_3^+ L_{h,c}$. The other results of this kind are contained in Section 5.

2. COMPUTING $H_*(\mathscr{L}_3; L_{h,c})$

The following is the main tool of the computation.

PROPOSITION 2.1. (cf. [4]). *Let*

$$\pi : U(\mathscr{L}_1) \to \mathbb{C}[e_1, e_2]$$

be the projection of the algebra $U(\mathscr{L}_1)$ onto its quotient over the two-sided ideal generated by e_3. Set

$$Q_{m,n}(t) = e_1^2 + \alpha_{m,n}(t)\, e_2,$$

where $\alpha_{m,n}(t) = m^2 t + 2mn + n^2 t^{-1}$. Then

$$\pi(\sigma_{k,l}(t))^2 = \prod_{\substack{0<u<k \\ 0<v<l}} Q_{k-l-2u,l-1-2v}(t). \qquad \blacksquare$$

REMARKS 1. Since $Q_{m,n}(t) = Q_{-m,-n}(t)$, then in the right hand side of the last formula each factor occurs twice, with the possible exception of $Q_{0,0} = e_1^2$. Hence the square root from the right hand side can be extracted explicitly.

2. If $t = -q/p$, which corresponds to $c = c_{p,q}$, then

$$\alpha_{m,n}(t) = (c_{p,q} - 1 - 24h_{m,n})/6.$$

The expression $c - 1 - 24\,h$ is of great importance in the representation theory of the Virasoro algebra; in particular, the line $c - 1 - 24\,h = 0$ is a common tangent to all the second order curves determined by the Kac equations.

Since Verma modules are free over $\mathscr{L}_1$, the results of Section 1 yield for the $\mathscr{L}_1$-module $L_{h,c}$ a free resolution of one of the forms

$$0 \leftarrow L_{h,c} \leftarrow V \leftarrow \ldots \leftarrow V \leftarrow 0$$

$$0 \leftarrow L_{h,c} \leftarrow V \leftarrow V \leftarrow V \leftarrow \ldots$$

$$0 \leftarrow L_{h,c} \leftarrow V \leftarrow V \oplus V \leftarrow \ldots \leftarrow V \oplus V \leftarrow V \leftarrow 0$$

$$0 \leftarrow L_{h,c} \leftarrow V \leftarrow V \oplus V \leftarrow V \oplus V \leftarrow V \oplus V \leftarrow \ldots$$

(compare the diagrams (1) – (5) in Section 1), where V denotes the free $\mathscr{L}_1$-module with one generator, and the arrows (besides two left ones in each row) are the homomorphisms $\varphi_{m,n}(t)$ or their sums (with appropriate signs). These resolutions are free $\mathscr{L}_3$-resolutions as well. Hence the homology of the Lie algebra $\mathscr{L}_3$ with the coefficients in $L_{h,c}$ may be calculated (as $\mathbb{C}\,[e_1, e_2]$-modules) as the homology of one of complexes

$$P \leftarrow P \leftarrow \ldots \leftarrow \begin{cases} 0 \\ P \leftarrow \ldots \end{cases}$$

$$P \leftarrow P \oplus P \leftarrow \ldots \leftarrow P \oplus P \leftarrow \begin{cases} P \leftarrow 0 \\ P \oplus P \leftarrow \ldots \end{cases}$$

where $P = \mathbb{C}\,[e_1, e_2]$, and the arrows denote P-linear mappings with matrix entries $\pi(\sigma_{k,l}(t))$ (see Proposition 2.1).

This homology is easy to find. To do this we put (for p, q, m, n fixed) for integers a, b, c, d such that $a \equiv b \bmod 2$, $c \equiv d \bmod 2$

$$R\{a, b; c, d\}(t) = \prod_{\substack{a<\alpha<b,\alpha\equiv a-1 \bmod 2 \\ c<\beta<d,\beta\equiv c-1 \bmod 2}} Q_{\alpha\beta}(t)$$

and set

$$A_{2i} = R\{2ip - m, 2ip + m; -n, n\},$$

$$A_{2i+1} = R\{2ip + m, (2i+2)p - m; -n, n\};$$

$$B_{2i} = R\{(2i-1)p + m, (2i+1)p - m; -q + n, q - n\},$$

$$B_{2i+1} = R\{(2i+1)p - m, (2i+1)p + m; -q + n, q - n\};$$

$$C_{2i} = R\{-m, m; 2ip - n, 2iq + n\},$$

$$C_{2i+1} = R\{-m, m; 2iq + n, (2i+2)q - n\};$$

$$D_{2i} = R\{-p + m; (2i-1)q + n, (2i+1)q - n\},$$

$$D_{2i+1} = R\{-p + m, p - m; (2i+1)q - n, (2i+1)q + n\}.$$

Thus all A, B, C, D are polynomial s in e_1, e_2 with coefficients being functions of t. Remark that $A_0 = C_0$ and $B_0 = D_0$ and that the square root may be extracted from these two polynomials, that is $A_0 = a_0^2$ and $B_0 = b_0^2$ (compare Remark 1 to Proposition 2.1).

We are interested in the case $t = -q/p$ (and also $t = q/p$). Remark that

$$Q_{\alpha,\beta}\left(\pm\frac{q}{p}\right) = Q_{\alpha+p,\beta\mp q}\left(\pm\frac{q}{p}\right).$$

This equality, together with $A_{\alpha,\beta}(t) = Q_{-\alpha,-c}(t)$, implies the following result.

LEMMA. *If $t = \pm q/p$, then*

$$C_{2i} = A_{2i},\ C_{2i+1} = B_{2i+1},\ D_{2i} = B_{2i},\ D_{2i+1} = A_{2i+1}.$$

All other pairs of polynomials A, B, C, D with $t = -q/p$ are prime to each other, and the same is true for $t = q/p$ ∎

(The last assertion is evident, for all our polynomials are decomposed into the factors of the form $e_1^2 + \alpha e_2$, and one has only to compare the sets of α's involved. By the way, it is important that p and q cannot be both even).

The results of Section 1 and the previous Lemma show that the homology of the Lie algebra $\mathscr{L}_3$ with the coefficients in the module $L_{h,c}$ is calculated as the homology of the complex exhibited at Fig. 2 (where P denotes $\mathbb{C}[e_1, e_2]$ and a_0, b_0, A_i, B_i denote $a_0(-q/p)$, $b_0(-q/p)$, $A_i(-q/p)$, $B_i(-q/p)$, and arrow

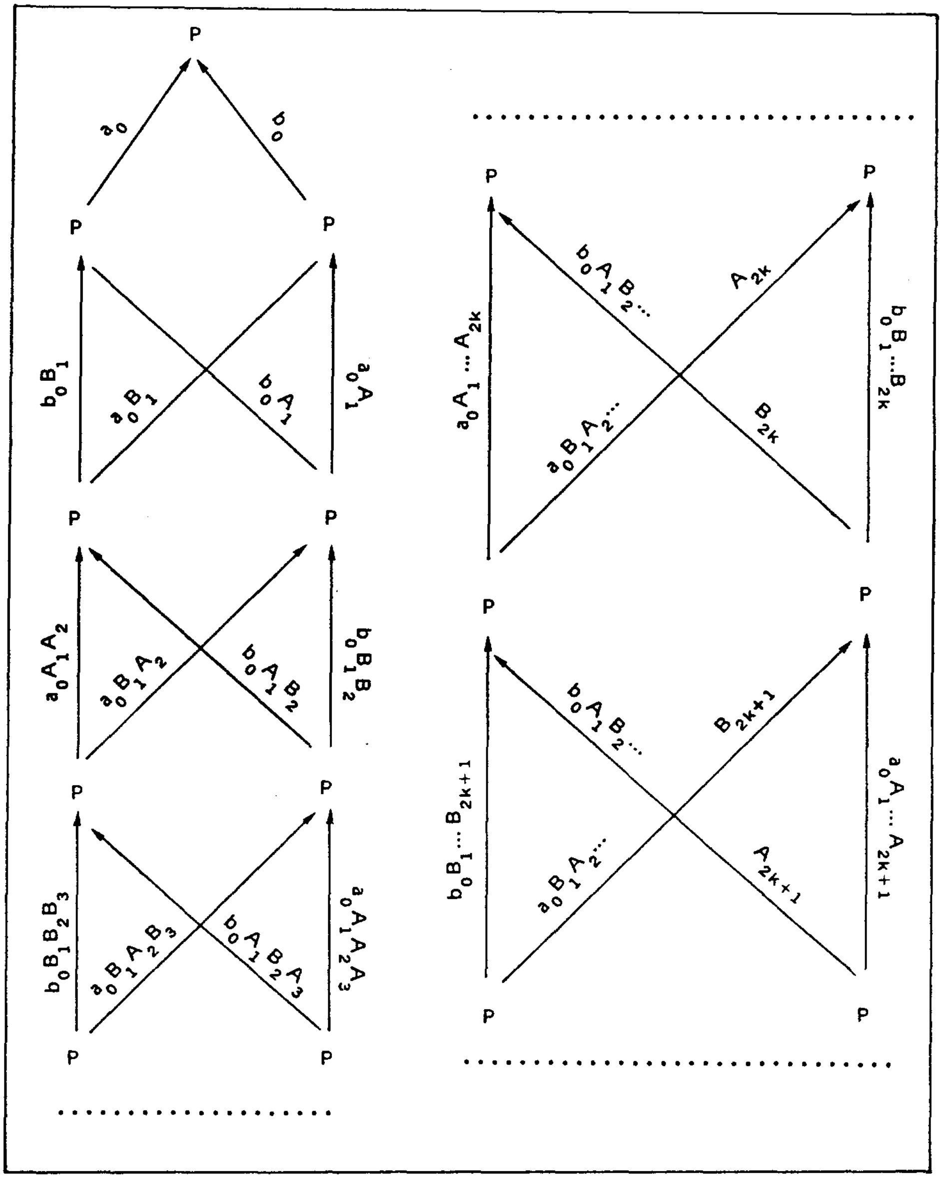


Fig.2

labelled with a polynomial denote the ± multiplication by this polynomial). A very simple calculation leads to the following result.

THEOREM 2.2. *As a* $\mathbb{C}[e_1, e_2]$*-module*

$$H_r(\mathscr{L}_3; L_{c_{p,q},h_{m,n}}) = $$
$$= \begin{cases} \mathbb{C}[e_1, e_2]/(a_0A_2 \ldots A_{2k}, \ b_0B_2 \ldots B_{2k}) \text{ for } r = 2k, \\ \mathbb{C}[e_1, e_2]/(A_1A_3 \ldots A_{2k+1}, \ B_1B_3 \ldots B_{2k+1}) \text{ for } r = 2k+1. \end{cases}$$

It is seen from this result that all the homologies of Theorem 2.2 are finite-dimensional. It is easy to find their dimensions, the results is as follows.

COROLLARY 2.3.

$$\dim H_r(\mathscr{L}_3; L_{h_{m,n},c_{p,q}}) = (r+1)^2 \quad \frac{mn(p-m)(q-n)}{2}$$

(In the case of trivial one-dimensional module $L_{0,0}$ corresponding to $p = 3$, $q = 2, m = n = 1$, this assertion is well known, cf. [6]).

If the module $L_{h,c}$ enters a diagram of the form (3) but is not its vertex, then the homology $H_*(\mathscr{L}_3; L_{h,c})$ is calculated with the use of the complex which may be obtained from Fig. 2 by an appropriate truncation: one takes a non-top P and preserves all the arrows which lead to this P. It is easy to see that the polynomials attached to the two arrows directed to any non-top P have a non-trivial common factor; therefore dim $H_0(\mathscr{L}_3; L_{h,c}) = \infty$. In the rest homology remains unchanged (in particular, it remains finite-dimensional): if $V_{h',c}$ enters the s-th term of the resolution of a quasi-finite module $L_{h,c}$, then for $r > 0$

$$H_r(\mathscr{L}_3; L_{h',c}) = H_{r+s}(\mathscr{L}_3; L_{h,c}).$$

Considering all possible cases we see that for all other cases the homology, and in particular, 0-dimensional homology, is infinite-dimensional. For example, if $V_{h,c}$ enters a diagram of the form (5) then the complex for computing $H_*(\mathscr{L}_3; L_{h,c})$ can be obtained from the complex on Fig. 2 by removing a non-top P and all the arrows which lead to this P; the homology will be infinite-dimensional in two dimensions, including 0. Since 0-dimensional homology is the co-invariant space, we obtain the assertion which was mentioned in Section 1:

COROLLARY 2.4. *The space of $\mathscr{L}_3$-co-invariants of the module $L_{h,c}$ is finite-dimensional if and only if the module $L_{h,c}$ is quasi-finite.* ■

This Corollary motivates the following extension of the notion of quasi-finite module.

DEFINITION. *A category $\mathcal{O}$ Vir-module M is called quasi-finite if* dim

$H_0(\mathscr{L}_3; M) < \infty$. ■

THEOREM 2.5. *A category* $\mathcal{O}$ *module* M *is quasi-finite if and only if it is a finite sum of quasi-finite irreducible modules.*

Proof. Let M^0 be the contragradient module for M. Then M^0 may have only a finite number of singular vectors (because $\dim H_0(\mathscr{L}_1; M) < \dim H_0(\mathscr{L}_3; M) < \infty$). Let v be one of them (modules from $\mathcal{O}$ necessarily have singular vectors), and let L be a submodule of M^0 generated by v. This is a quotient of a Verma module. Since the number of singular vectors in L is finite, L has a minimal submodule S; clearly, S is irreducible. The module $S = S^0$ is a quotient of M, hence $\dim H_0(\mathscr{L}_3; S) < \infty$, and S is quasi-finite, in particular $\dim H_1(\mathscr{L}_3; S) < \infty$. Let $S = M/M_1$. From the exact sequence

$$H_1(\mathscr{L}_3; S) \to H_0(\mathscr{L}_3; M_1) \to H_0(\mathscr{L}_3; M)$$

we see that $\dim H_0(\mathscr{L}_3; M_1) < \infty$. Thus the previous arguments are applicable to M_1. In the same time M_1^0 has less singular vectors than M^0; therefore we obtain a finite procedure producing a filtration

$$0 = M_N \subset \ldots \subset M_2 \subset M_1 \subset M_0 = M$$

with irreducible quasi-finite subsequent quotients M_i/M_{i+1}. Now to finish the proof we have only to prove the following

LEMMA *Let* S, S' *be irreducible quasi-finite modules. Then* $\mathrm{Ext}^1(S, S') = 0$

Proof. We have $\mathrm{Ext}^1(S, S') = H^1(\mathrm{Vir}, \mathbb{C}e_0; S^* \otimes S')$. To calculate this cohomology we tensor the BGG resolution for S' and the dual BGG resolution for S. We get a spectral sequence whose initial term is the sum of cohomologies of the form $H^r(\mathrm{Vir}, \mathbb{C}e_0; (V^0_{h',c})^* \otimes V_{h,c})$ which is non-trivial only for $h' = h$ and $r = 0$. This implies that $H^r(\mathrm{Vir}, \mathbb{C}e_0; S^* \otimes S')$ may be non-trivial only for r even (and $S' = S$). QED ▪ ■

REMARK. For an irreducible representation L of Vir we denote by $\mathscr{M}_L$ a class of such irreducible P that $\mathrm{Ext}^i(P, L) \neq 0$ for some i. Then denote by $\mathscr{K}_L$ the category of representations of Vir with all irreducible composition factors belonging to $\mathscr{M}_L$. One can easily deduce from above that $\mathscr{K}_L$ is isomorphic to $\mathscr{K}_{\mathbb{C}}$ if and only if L is quasi-finite.

3. COMPUTING $H_*(\mathscr{L}_{1,1}; L_{h,c})$

The Virasoro algebra Vir can be naturally interpreted as the central extension of the Lie algebra of meromorphic vector fields in the line of the form $z^{-N} p(z)\, d/dz$, where N is an integer and p is a polynomial. In this interpretation $\mathscr{L}_k$ is the Lie algebra of vector fields of the form $z^{k+1}p(z)\, d/dz$. We define $\mathscr{L}_{1,1}$ as the Lie algebra of vector fields of the form $z^2(z-1)^2\, d/dz$. This Lie algebra is obtained from $\mathscr{L}_3$ by a deformation inside $\mathscr{L}_1$. We show in this Section that the homology with the coefficients in $L_{h,c}$ does not change under this deformation.

THEOREM 3.1. $H_*(\mathscr{L}_{1,1}; L_{h,c}) \simeq H_*(\mathscr{L}_3; L_{h,c})$.

We make this statement more precise. The Lie algebra $\mathscr{L}_{1,1}$ is a codimension 2 ideal in the Lie algebra $\mathscr{L}_{0,0}$ of vector fields of the form $z(z-1)\,p(z)\, d/dz$ (which is, in turn, a deformation of $\mathscr{L}_1$). The quotient $\mathscr{L}_{0,0}/\mathscr{L}_{1,1}$ is generated by the vector fields $e_0' = z(z-1)^2\, d/dz$ and $e_0'' = z^2(z-1)\, d/dz$, and these vector fields define a pair of commuting operators in $H_*(\mathscr{L}_{1,1}; L_{h,c})$. Thus $H_*(\mathscr{L}_{1,1}; L_{h,c})$, as well as $H_*(\mathscr{L}_3; L_{h,c})$, is a module over the algebra of polynomials in two variables, but there is an essential difference between the two modules; the operators e_0', e_0'' in $H_*(\mathscr{L}_{1,1}; L_{h,c})$ are diagonizable, while the operators e_1, e_2 in $H_*(\mathscr{L}_3; L_{h,c})$ are (virtually) nilpotent.

Describe the situation for a quasi-finite $L_{h,c}$ in more details.

THEOREM 3.2. *The operators e_0', e_0'' in $H_r\ (\mathscr{L}_{1,1}; L_{h_{m,n}, c_{p,q}})$ have simple spectra; all the eigenvalues are of the form $h_{m',n'}$ (for the same p,q), but not necessarily with $0 < m' < p$, $0 < n' < q$.* ■

The results of this Section contain a complete description of the spectra of e_0', e_0''. Now we give the precise statement only for the 0-dimensional homology, that is for the co-invariants.

An ordered triple of pairs of integers (m, n), (m', n'), (m'', n'') is called admissible if $0 < m < p$, $0 < m' < p$, $0 < m'' < p$, $0 < n < q$, $0 < n' < q$, $0 < n'' < q$, $p < m + m' + m'' < 2p$, $q < n + n' + n'' < 2p$, and the sums $m + m' + m''$, $n + n' + n''$ are odd.

THEOREM 3.3. *Let $\{(m, n), (m_i', n_i'), (m_i'', n_i'') \mid i = 1, \dots, N\}$ be the set of all admissible triples with the fixed first pair (m, n). Then $H_0(\mathscr{L}_{1,1}; L_{h_{m,n}, c_{p,c}})$ can be decomposed into the sum of N one-dimensional spaces, such that in the i-th one of these spaces e_0' and e_0'' coincide with the multiplication by $h_{m_i',n_i'}$ and $h_{m_i'',n_i''}$*

Now we turn to the proofs.

Using the notations $e_k = z^{k+1}\, d/dz$ we have $e_0' = e_2 - 2e_1 + e_0$ and $e_0'' = e_2 - e_1$. The Lie algebra $\mathscr{L}_{1,1}$ is generated by the vector fields $e_3 - 2e_2 + e_1$, $e_4 - 2e_3 + e_2$, . . . Evidently $[e_0', e_0''] \in \mathscr{L}_{1,1}$. Verma modules $V_{h,c}$ are free not only over $\mathscr{L}_1$, but also over $\mathscr{L}_{0,0}$, and, certainly, over $\mathscr{L}_{1,1}$. Hence the resolutions of the modules $L_{h,c}$ constructed in Section 1 are $\mathscr{L}_{1,1}$-free resolutions as well, and the homology $H_*(\mathscr{L}_{1,1}; L_{h,c})$ is just the homology of corresponding $\mathscr{L}_{1,1}$-co-invariants complexes. The $\mathscr{L}_{1,1}$-co-invariants space in the Verma module $V_{h,c}$ is a free module with one generator over the ring $\mathbb{C}[e_0', e_0'']$, so the complexes look quite like the complexes from Section 2, only with P being not $\mathbb{C}[e_1, e_2]$ but $\mathbb{C}[e_0', e_0'']$.

The differentials of our co-invariant complex are composed from the mappings $P \to P$ obtained from the inclusions $V_{h',c} \to V_{h,c}$ by passing to co-invariants. This mapping is nothing but the multiplication by the polynomial in e_0', e_0'' which is the projection of the operator $\sigma_{k,l}(t)$, determining the singular vector of the module $V_{h,c}$ (the image of $v_{h',c}$) onto $\mathbb{C}[e_0', e_0'']$.

This projection has been actually found in the paper [4]. Namely, this paper contains explicit formulas describing the action of the operators $\sigma_{k,l}(t)$ in the modules $\mathscr{F}_{\lambda,\mu}$ of polynomial tensor densities on the circle. More exactly, $\mathscr{F}_{\lambda,\mu}$ has a $\mathbb{C}$-base f_j ($j \in \mathbb{Z}$), and the action of Vir in $\mathscr{F}_{\lambda,\mu}$ is given by the formulas $e_i f_j = (\mu + j - \lambda(i+1)) f_{i+j}$, $z f_j = 0$. In particular, $\sigma_{k,l}(t) f_0 = P(\lambda, \mu, k, l, t) f_{kl}$. In [4] an explicit formula for $P(\lambda, \mu, k, l, t)$ is given:

$$P(\lambda, \mu, k, l, t) = \prod_{\substack{0<u<k \\ 0<v<l}} R_{k,l,u,v}(t; \lambda, \mu),$$

$$\begin{aligned} R_{k,l,u,v}(t;\lambda,\mu) &= (\mu - 2\lambda)^2 + \\ &+ (\mu - 2\lambda)\,[(2u(k-1-u) + k - 1)\,t + kl - (k-1-2u)(l-1-2v) - 1 + \\ &+ (2v(l-1-v) + l - 1)\,t^{-1}] - \\ &- \lambda[(k-1-2u)^2 t + 2(k-1-2u)(l-1-2v) + (l-1-2v)^2 t^{-1}] + \\ &+ \frac{(ut+v)((u+1)t + (v+1))((k-u)t + (l-v))((k-1-u)t + (l-1-v))}{t^2}. \end{aligned}$$

Our situation is related to the modules $\mathscr{F}_{\lambda,\mu}$ in the following way. Consider in the $\mathscr{L}_{1,1}$-co-invariants space of the module $V_{h,c}$ the eigenspace of the operators e_0', e_0'' corresponding to the eigenvalues α and β. Let F be a functional on this space; it may be regarded as a sequence of functionals $F_j : (V_{h,c})_j \to \mathbb{C}$ (the last subscript j corresponds to the natural grading in $V_{h,c}$). The functional

F must satisfy the conditions $e_0' F = \alpha F$, $e_0'' F = \beta F$, $(e_3 - 2e_2 + e_1) F = 0$, $(e_4 - 2e_3 + e_2) F = 0, \ldots$; these conditions are evidently equivalent to the condition

$$e_i F_{i+j} = (h + j - \alpha + i\beta) F_j,$$

that is $e_i F_{i+j} = (\mu + j - \lambda(i + 1)) F_j$ with $\mu = h - \beta - \alpha$, $\lambda = -\beta$.

In particular, our reduced co-invariant space is one-dimensional, and our mapping restricted to these space is the multiplication by a number (which is the value of the polynomial in e_0', e_0'' to be find for $e_0' = \alpha$, $e_0'' = \beta$). Let us find this number taking for the base element of our dual co-invariant space the functional F normed by the condition $F(v) = 1$. Our mapping $V_{h',c} \to V_{h,c}$ takes $v' = v_{h',c}$ into $\sigma_{k,l}(t)\, v$. Hence the adjoint mappings takes the functional F into the functional F' whose value at the point v' is

$$F'(v') = F(\sigma_{k,l}(t)v) = P(\lambda, \mu, k, l, t)\, F(v) = P(\lambda, \mu, k, l, t).$$

Having this in mind we can easily calculate the homology of the Lie algebra $\mathscr{L}_{1,1}$ with the coefficients in any module $L_{h,c}$. We consider in details only the most interesting case of quasi-finite module $L_{h_{m,n}, c_{p,q}}$.

Set

$$\lambda = -e_0'', \quad \mu = h_{m,n} - e_0' - e_0'', \quad t = -\frac{q}{p}.$$

$P(\lambda, \mu, m, n, t)$ becomes the square root from

$$\prod_{\substack{0<u<m \\ 0<v<n}} R_{m,n;u,v}(e_0', e_0''),$$

where

$$R_{m,m;u,v}(e_0', e_0'') = (e_0'' - e_0')^2 - \frac{[(m - 2u - 1)\, q - (n - 2v - 1)p]^2}{2pq} (e_0' + e_0'') + $$
$$+ h_{m-2u, n-2v} h_{n-2(u+1), n-2(v+1)}.$$

Notice that the equation $R_{m,n;u,v}(\xi, \eta) = 0$ defines a parabola in the plane (ξ, η), and that the parabolas $R_{m,n;u,v}(\xi, \eta) = 0$ and $R_{m',n'u',v'}(\xi, \eta) = 0$ have two common points;

$$\left(h_{\frac{m+m'}{2}-(u+u'+1),\frac{n+n'}{2}-(v+v'+1)},\ h_{\frac{m-m'}{2}-u+u',\frac{n-n'}{2}-v+v'}\right)$$

and the point which is obtained from the indicated one by the permutation of the coordinates. (The subscripts may fail to be integrals, but actually $h_{a,b}$ depends only on $qa - pb$ which is an integer in all cases considered below).

Our co-invariant complex is arranged as follows. One should take Fig. 1, replace each dot with the letter P (having in mind that $P = \mathbb{C}[e_0', e_0'']$) and regarded an arrow with the numbers a, b attached as the multiplication by the polynomial $\prod_{\substack{0<u<a\\0<v<b}} R_{a,b;u,v}$. Now we are able to find the $\mathbb{C}[e_0', e_0'']$-modules $H_r(\mathscr{L}_{1,1}; L_{h_{m,n}, c_{p,q}})$. The final result is as follows.

The space $H_r(\mathscr{L}_{1,1}; L_{h_{m,n}, c_{p,q}})$ has dimension

$$\frac{(r+1)^2 mn(p-m)(q-n)}{2}$$

and decomposes into the sum of one-dimensional eigenspaces of e_0' and e_0''. The eigenvalues are the coordinates of the intersection points of parabolas;

For $r = 2s$

$$R_{2sp+m,n;u,v}(\xi, \eta) = 0$$

and

$$R_{p-m,(2s+1)q-n;u',v'}(\xi, \eta) = 0$$

($xp \leqslant u < xp + m$, $x = 0, 1, \ldots, 2s$, $0 \leqslant v < n$; $o \leqslant u' < p - m$, $yq \leqslant v' < (y+1)q - n$, $y = 0, 1, \ldots, 2s$);

for $r = 2s + 1$

$$R_{(2s+1)p+m,q-n;u,v}(\xi, \eta) = 0$$

and

$$R_{p-m,(2s+1)q+n,u',v'}(\xi, \eta) = 0$$

($xp \leqslant u < xp + m$ $x = 0, 1, \ldots, 2s+1$, $0 \leqslant v < q - n$; $0 \leqslant u < p - m$, $yq \leqslant v' < yq + n$, $y = 0, 1, \ldots, 2s+1$).

(Each of the above families of parabolas involve each parabola twice; the coordinates of the intersection points of the parabolas are indicated above).

To finish this Section we consider the limit pass which converts the homology of the Lie algebra $\mathscr{L}_{1,1}$ into that of the Lie algebra $\mathscr{L}_3$. The Lie algebra $\mathscr{L}_{1,1}$ may be realized as the Lie algebra of vector fields of the form $z^2(z -$

$-\epsilon)^2 p(z)d/dz$, where $0 \neq \epsilon \in \mathbb{C}$. When $\epsilon = 0$ this Lie algebra becomes $\mathscr{L}_3$. The previous calculation changes a little under this interpretation. Namely, now we have $e_0' = e_2 - 2\epsilon e_1 + \epsilon^2 e_0$, $e_0'' = e_2 - \epsilon e_1$, and $\mathscr{L}_{1,1}$ is generated by the fields $e_3 - 2\epsilon e_2 + \epsilon^2 e_1$, $e_4 - 2\epsilon e_3 + \epsilon^2 e_2, \ldots$; the formula for $e_i F_{i+j}$ takes the form

$$e_i F_{i+j} = [(i\beta - \alpha)\ \epsilon^{i-2} + (h+j)\epsilon^i]\, F_j,$$

that is

$$e_i \frac{F_{i+j}}{\epsilon^{i+j}} = \left[\frac{i\beta-\alpha}{\epsilon^2} + h + j\right]\frac{F_j}{\epsilon^j} = (\mu + j - \lambda(i+1))\ \frac{F_j}{\epsilon^j}\ ,$$

where

$$\mu = h - \frac{\alpha+\beta}{\epsilon^2}, \quad \lambda = -\ \frac{\beta}{\epsilon^2}$$

The mapping

$$\mathbb{C}[e_0', e_0''] \ \rightarrow\ \mathbb{C}[e_0', e_0'']$$

which is the quotient of the mapping $V_{h',c} \rightarrow V_{h,c}$ corresponding to the singular vector $\sigma_{k,l}(t)v \in V_{h,c}$ is the multiplication by the polynomial obtained from $P(\lambda, \mu, k, l, t)$ by the subsitution

$$\mu = h - \frac{e_0' + e_0''}{\epsilon^2}, \qquad \lambda = -\ \frac{e_0''}{\epsilon^2}$$

and the multiplication by ϵ^{kl}. This suggests that the formuls for $(\sigma_{k,l}(t))$ from Section 2 must be obtained in the limit from the formula for $P(\lambda, \mu, k, l, t)$; and really

$$\lim_{\epsilon\rightarrow 0} \epsilon^2 R_{m,n;u,v}\left(t;\ -\ \frac{e_2 - \epsilon e_1}{\epsilon^2},\ -\ \frac{2e_2 - 3\epsilon e_1}{\epsilon^2}\right) =$$

$$= e_1^2 + [(k-1-2u)^2 t + 2(k-1-2u)(l-1-2v)^2]\, e_2 =$$

$$= Q_{k-1-2u,\ l-1-2v}(t)$$

(see Section 2). ■

4. THE HOMOLOGY OF NILPOTENT SUBALGEBRAS OF THE KAC-MOODY ALGEBRAS

For the general information on Kac-Moody Lie algebras see Kac's book [7].

It is generally accepted that a proper analog of a finite-dimensional representation of a semisimple Lie algebra in the Kac-Moody theory is a representation with a (regular) integral dominant highest weight. And it is exactly this class of modules for which the results similar to those of Section 2 are valid. The role of $\mathcal{L}_3$ plays the commutant $[\mathfrak{n}_+, \mathfrak{n}_+]$ of the nilpotent Lie algebra $\mathfrak{n}_+$ from the Cartan decomposition of the Kac-Moody algebra $\mathfrak{g}$.

THEOREM 4.1. *Let $\mathfrak{g}$ be a Kac-Moody Lie algebra with a symmetrizable Cartan matrix, and let L_λ be an irreducible $\mathfrak{g}$-module with a regular integral dominant highest weight $\lambda = (\lambda_1, \ldots, \lambda_l)$, where $l = \operatorname{rank} \mathfrak{g}$. Then $\dim H_0([\mathfrak{n}_+, \mathfrak{n}_+]; L_\lambda) = (\lambda_1 + 1) \ldots (\lambda_2 + 1)$.*

Proof. It is known that L_λ has a Bernstein-Gelfand-Gelfand resolution composed of Verma modules labelled with the elements of the Weyl group (see [12]). The space of $[\mathfrak{n}_+, \mathfrak{n}_+]$-co-invariant in a Verma module is $\mathbb{C}[e_1, \ldots, e_l]$ where $e_1, \ldots, e_l$ are the standard generators of $\mathfrak{n}_+$. Thus $H_*([\mathfrak{n}_+, \mathfrak{n}_+]; L_\lambda)$ can be calculated from the complex

$$P \leftarrow P^{m_1} \leftarrow P^{m_2} \leftarrow \ldots$$

where $P = \mathbb{C}[e_1, \ldots, e_l]$ and m_i is the number of height i elements in the Weyl group. In particular, $m_1 = l$, and the left arrow is the mapping $(p_1, \ldots, p_l) \to e_1^{\lambda_1+1} p_1 + \ldots + e_l^{\lambda_l+1} p_l$, whence the result. ■

The BBG resolution of this proof allows to calculate higher homology as well. In particular, one easily gets

THEOREM 4.2. *If $\operatorname{rank} \mathfrak{g} = 2$, then*

$$\dim H_r([\mathfrak{n}_+, \mathfrak{n}_+]; L_\lambda) = (r+1)^2 (\lambda_1 + 1) \ldots (\lambda_1 + 1).$$

Moreover, as a $\mathbb{C}[e_1, e_2]$-module $H_r([\mathfrak{n}_+, \mathfrak{n}_+]; L) =$

$$= \begin{cases} \mathbb{C}[e_1, e_2]/(e_1^{(r+1)(\lambda_1+1)}, e_2^{(r+1)(\lambda_2+1)}) & \text{if } r \text{ is even,} \\ \mathbb{C}[e_1, e_2]/(e_1^{(r+1)(\lambda_2+1)}, e_2^{(r+1)(\lambda_1+1)}) & \text{if } r \text{ is odd.} \end{cases}$$ ■

The details of the proof are left to the reader.

In the general case we restrict ourselves to

CONJECTURE 4.3. *In the situation of Theorem 4.1*

$$\dim H_r([\mathfrak{n}_+, \mathfrak{n}_+]; L_\lambda) < \infty$$

for all r . ■

As to other highest weight irreducible modules, there is a natural

CONJECTURE 4.4. *If* λ *is not a regular integral dominant weight then* $\dim H_0([\mathfrak{n}_+, \mathfrak{n}_+]; L_\lambda) = \infty$.

We can prove this Conjecture in two cases. In the case of rank 2 the assertion can be easily deduced from structural results of Malikov [9]. The second case is when $\mathfrak{g}$ is an extended current algebra, and λ does not belong to the imaginary root hyperplane (see [10]). In this case the Verma module V_λ has some (at most l) independent singular vectors. If the maximal submodule of V_λ is generated by these vectors, then the proof is easy: these vectors become, after reducing modulo $[\mathfrak{n}_+, \mathfrak{n}_+]$, monomials of the form $e_1^{\alpha_1} \dots e_l^{\alpha_l}$, these monomials have a non-trivial common factor (if the weight is not integral dominant) and generate a codimension infinity ideal in $\mathbb{C}[e_1, \dots, e_l]$. The trouble is that the singular vectors generate only a submodule M' of the maximal submodule $M \subset V_\lambda$, M/M' has singular vectors of its own, and so on. But all these secondary, ternary, , , , , singular vectors have weights of singular vectors of V_λ and therefore corresponding monomials again have the same factor.

Finally note that Remark in the end of Section 2 remains valid only partially. For example, let L be an irreducible heighest weight representation of the Kac-Moody Lie algebra $\hat{\mathfrak{sl}}_2$. Form the category $\mathcal{K}_L$ exactly as in Remark in Section 2. If L corresponds to an integral dominnat weight, then $\mathcal{K}_L \cong \mathcal{K}_{\mathbb{C}}$. But the inverse fails to be true: there exists some L with $\mathcal{K}_L \cong \mathcal{K}_{\mathbb{C}}$ and L is not integral dominant. Such representations play an essential role in Polyakov's quantum gravity theory [8].

5. COMMENTS, CONJECTURES, AND ANNOUNCEMENTS

1 Let E be a smooth compact complex curve and Lie (E) be the Lie algebra of meromorphic vector fields on E. Each point $y \in E$ determines a two-dimensional cohomology class ω_y of Lie (E) with trivial coefficients. If a local coordinate z is choosen in a neighbourhood of y, $z(y) = 0$, then ω_y is represented by the cocycle

$$u_1, u_2 \to \mathrm{Res}_0 (f''g' - f'g'')\, dz/z$$

where $u_1, u_2 \in$ Lie(E), and $u_1 = f(z)\, d/dz$, $u_2 = g(z)\, d/dz$ in a neighbourhood of y.

The point $y \in E$ defines the adic topology in Lie (E) let $\mathcal{L}_y(E)$ be the completion of Lie (E) with respect to this topology. The formula (2) defines a 2-

-cocycle of the Lie algebra $\mathscr{L}_y(E)$. Let $\mathscr{L}(E) = \overline{\Pi}_y \mathscr{L}_y(E)$. Here the sign Π denotes the restricted direct product, that is $\mathscr{L}(E)$ is generated by the sets $\{l_y\}$ where $l_y \in \mathscr{L}_y(E)$, and only a finite number of $\{l_y\}$ may have a pole (at y). It is clear that $\omega = \Sigma_y \omega_y$ is a well defined cohomology class of the Lie algebra $\mathscr{L}(E)$. The corresponding central extension of $\mathscr{L}(E)$ is denoted by Vir (E). The Lie algebra Lie (E) is naturally embedded in $\mathscr{L}(E)$. The residue formula implies that the restriction of the class ω to Lie (E) vanishes. Therefore the embedding Lie $(E) \to \mathscr{L}(E)$ can be lifted to the embedding Lie $(E) \to$ Vir (E).

There is a technically more convenient version of this construction which involves not all the points of E but a finite set if them. Let $p_1, \ldots p_s$ be a set of points of E and $U = E - p_1, \ldots, p_s$. We assume usually that E is an affine manifold. Let Lie (U) be the subalgebra of Lie (E) consisting of those vector fields which have no poles in U. Put $\mathscr{L}(p_1, \ldots, p_s) = \oplus \mathscr{L}_{p_i}(E)$ and let Vir $(p_1, \ldots, p_s)$ be the central extension of the Lie algebra $\mathscr{L}(p_1, \ldots, p_s)$ corresponding to the class $\omega_{p_1} + \ldots + \omega_{p_s}$. It follows again from the residue formula that the restriction of the class $\omega_{p_1} + \ldots + \omega_{p_s}$ to the Lie algebra Lie (U) which is naturally embedded in $\mathscr{L}(p_1, \ldots, p_s)$ vanishes. This means that there is an injection Lie $(U) \to$ Vir $(p_1, \ldots, p_s)$.

The mathematical apparatus of the conformal field theory is similar in some respects to the representation theory of adele groups. We give here some necessary definitions.

Let for each point $y \in E$ a quasi-finite representation My of the Lie algebra Vir (y) is given, such that "the central charge" c of M_y is the same for all y, and that for all y but a finite number of them $M_y = L_{0,c}$. In the space $\overline{\oplus}_y M_y$ spanned by the sets $\{u_y \in M_y \mid y \in M\}$ with almost all u_y being vacuum vectors in M_y the Lie algebra Vir (E) acts naturally. It is natural to call a finite sum of such modules a quasi-finite module over Vir (E).

Let M be a quasi-finite module over Vir (E). Evidently the space H_0 (Lie (E); M) is a direct analog for the space of automorphic forms.

THEOREM 5.1. *For any quasi-finite module M over Vir (E) the space H_0 (Lie (E); M) is finite-dimensional.* ■

This theorem is proved in the work of A. Beilinson and the first author [1].

Let $p_1, \ldots, p_s$ be a set of distinct points of E, $U = E - \{p_1, \ldots, p_s\}$ and N be a quasi-finite representation of the Lie algebra Vir $(p_1, \ldots, p_s)$. We construct a quasi-finite representation $\bar{N}$ of Lie (E) placing $L_{0,c}$ at all points of E different from $p_1, \ldots, p_s$.

PROPOSITION 5.2. *H_0(Lie (U); $N) = H_0$(Lie (E); $\bar{N}$).* ■

This is a direct consequence of the Frobenius duality.

COROLLARY 5.3. *The space $H_0(Lie(U); N)$ is finite-dimensional* ∎

2. Let A_c be the free Abelian group generated by all irreducible quasi-finite Vir-modules with a given c. For a set of distinct points $p_1, \ldots, p_s \in E$ and a set of irreducible quasi-finite representations $L_1, \ldots, L_s$ consider the natural action of Vir $(p_1, \ldots, p_s)$ in the space $L = L_1 \otimes L_2 \otimes \ldots \otimes L_s$. The correspondence $L_1, \ldots, L_s \to \dim H_0(\mathrm{Lie}(U); L)$ defines a polylinear symmetric form $\varphi_g^s; \otimes^s A_c \to \mathbb{Z}$ (it depends only on the genus g of the curve E).

Consider the case $g = 0$. The form φ_0^2 defines a non-degenerate quadratic form on A_c. The natural base of A_c consisting of irreducible representations is ortogonal with respect to this form. We identify A_c with A_c^* by means of this form and convert φ_0^3 into the homomorphism $\varphi : A_c \otimes A_c \to A_c$. The homomorphism φ defines on A_c a commutative algebra structure. (This assertion is actually contained in [13]).

We shall call the triple of modules $L_{h,c}, L_{h',c}, L_{h'',c}$ admissible if $c = c_{p,q}$, $h = h_{m,n}$, $h' = h_{m',n}$ $h'' = h_{m'',n''}$ and the triple (m, n), (m', n'), (m'', n'') is admissible in the sense of Section 3.

THEOREM 5.4. *The number φ_0^3 $(L_{h,c}, L_{h',c}, L_{h'',c})$ is equal to 1 if the triple $(L_{h,c'}, L_{h'',c}, L_{h'',c})$ is admissible and is equal to 0 otherwise*

This is actually proved in Section 3. In fact, let the three chosen points in $\mathbb{CP}^1 = \mathbb{C} \cup \infty$ be 0, 1, ∞, and let $U = \mathbb{C} - \{0, 1\}$. The main result of section 3 may be stated as follows. Let $1_{h',h''}$ be the one dimensional representation of $\mathcal{L}_{0,0}$ with $e_0' = h'$, $e_0'' = h''$. Then $\dim H_0(\mathcal{L}_{0,0}; L_{h,c} \otimes 1_{h',h''}) = 1$ if $(L_{h,c}, L_{h',c}, L_{h'',c})$ is an admissible triple and $= 0$ otherwise. Then the $\mathcal{L}_{0,0}$-module $L_{h,c} \otimes 1_{h',h''}$ induce the Lie (U)-module $L_{h,c} \otimes V_{h',c} \otimes V_{h'',c}$, whence $H_0(\mathcal{L}_{0,0}; L_{h,c} \otimes 1_{h',h''}) = H_0(\mathrm{Lie}(U); L_{h,c} \otimes V_{h''} \otimes V_{h'',c})$ It remains to show that in the last homology $V_{h',c}$ and $V_{h'',c}$ may be replaced by $L_{h',c}$ and $L_{h'',c}$. But there is an exact sequence of the form $V_{\alpha,c} \oplus V_{\beta,c} \to V_{h',c} \to L_{h',c} \to 0$, and $H_0(\mathrm{Lie}\,(U); L_{h,c} \otimes (V_{\alpha,c} \oplus V_{\beta,c}) \otimes V_{h'',c}) = H_0(\mathcal{L}_{0,0}; L_{h,c} \otimes (1_{\alpha,h''} \oplus 1_{\beta,h''})) =$ $= 0$ bevause none of the triples $(L_{h,c}, L_{\alpha,c}, L_{h'',c})$, $(L_{h,c}, L_{\beta,c}, L_{h'',c})$ is admissible. Therefore $H_0(\mathrm{Lie}\,(U); L_{h,c} \otimes V_{h',c} \otimes V_{h'',c}) = H_0(\mathrm{Lie}\,(U); L_{h,c} \otimes L_{h',c} \otimes V_{h'',c})$, and the similar arguments hold for $V_{h'',c}$. QED

The operators φ_0^2 and φ_0^3 determine all other φ_g^p. Since the non-degenerate scalar product is fixed on A_c, there defined the operator $A_c \to A_c \otimes A_c$ dual to the multiplication. Denote by I the element of A_c corresponding to the representation $L_{0,c}$, and let S be the image of I under the composition of this co-multiplication $A_c \to A_c \otimes A_c$ and the multiplication $A_c \otimes A_c \to A_c$.

THEOREM 5.5. *The image of* $a_1 \otimes \ldots \otimes a_p$ *under the mapping* $\varphi_g^p \colon A_c \to \mathbb{Z}$ *is equal to* $\langle I, S^g a_1 a_2, \ldots, a_p \rangle$, *where* $\langle\ ,\ \rangle$ *is the scalar product in* A_c. ■

This statement is known to physicists for a long time; for a mathematical proof of it see [1].

3. In this subsection we exhibit a mathematical constructon corresponding to the operation of the creation of particles in the conformal fied theory. Let E be a complex curve, and $p_1, \ldots, p_s, q_1, \ldots, q_r$ be a set of distinct points in E. Let M be a quasi-finite representation of the Lie algebra Vir $(p_1, \ldots, p_s)$. Assume also that $U = E - \{p_1, \ldots, p_s\}$ is an affine manifold. Denote by $\mathrm{Lie}_N(U)$ the subalgebra of Lie (U) composed of field which have N-fold zeroes in the points $q_1, \ldots, q_r$. Let $M_N = H_0(\mathrm{Lie}_N(U); M)$. There is a natural mapping of M_N onto M_{N-1}. Consider the corresponding sequence of embeddings: $M_0^* \to M_1^* \to M_2^* \to \ldots$ The inductive limit $M^*(q_1, \ldots, q_r)$ is a Vir $(q_1, \ldots, q_r)$-module in a natural sense. In fact, if a vector field $f \in$ Lie $(U - \{q_1, \ldots, q_r\})$ has in the points $q_1, \ldots, q_r$ poles of order not greater than D, then f defines a mapping $M_i^* - M_{i+D+1}^*$. Hence f induces an operator in the inductive limit $M^*(q_1, \ldots, q_r)$. We get an action in $M^*(q_1, \ldots, q_r)$ of the completion of the Lie algebra Lie $(E - \{p_1, \ldots, p_s, q_1, \ldots, q_r\})$ with respect to the topology generated by the family of neighbourhoods $\mathrm{Lie}_N(U)$. It is easy to see that the corresponding representation will be projective, so we obtain on $M^*(q_1, \ldots, q_r)$ a structure of Vir $(q_1, \ldots, q_r)$-module. This module is quasi-finite.

Notice that there is a Lie $(E - \{p_1, \ldots, p_s, q_1, \ldots, q_r\})$-invariant form $M \otimes M^*(q_1, \ldots, q_r) \to \mathbb{C}$.

We have constructed a contavariant functor from the category of representations of the Lie algebra Vir $(p_1, \ldots, p_s)$ into the category of representations of the Lie algebra Vir $(q_1, \ldots, q_r)$. This gives us a mapping $\theta_g^{s,r} \colon \otimes^s A_c \to \otimes^r A_c$ which depends on the genus of the curve E.

THEOREM 5.6. *The scalar product on* A_c *converts the mapping* $\theta_g^{s,r}$ *into a form* $\otimes^{r+s} A_c \to \mathbb{Z}$, *which coincides with* φ_g^{r+s}. ■

This theorem allows us to find the module $M^*(q_1, \ldots, q_r)$ if the decomposition of M into irreducible components is known.

REMARK. The correspondence $M \to M^*$ is understood by physicists as the creation of particles related to irreducible quasi-finite representations. Particles situated in the points $p_1, \ldots, p_s$ create particles in the points $q_1, \ldots, q_r$. Let $E = \mathbb{C}P^1$, $s = 1, r = 2, p_1 = \infty, q_1 = 0, q_2 = 1$. Place to ∞ the representation $L_{0,c}$. Then $L^*_{0,c}$ $(0, 1) = \oplus(L_{h,c} \otimes L_{h,c})$ where summation is taken over all irreducible quasi-finite representations. Thus $L_{0,c}$ (the "vacuum representation") creates from itself all other quasi-finite representations for the given c.

4. Let $E = \mathbb{C}P^1$. M be a quasi-finite representation of the Lie algebra Vir (∞), $q_1, \ldots, q_r$ be a set of distinct points of $\mathbb{C} = \mathbb{C}P^1 - \infty$. Denote by $\mathcal{L}(q_1, \ldots, q_r)$ the subalgebra of the Lie algebra Lie (U) consisting of vector fields vanishing at the points $q_1, \ldots, q_r$ with the first derivative.

THEOREM 5.7. *There is a non-degenerate pairing between the spaces* $H_i(\mathcal{L}(q_1, \ldots, q_r); M)$ *and* $H^i(\mathcal{L}(q_1, \ldots, q_r); M^*(q_1, \ldots, q_r))$ *induced by the natural pairing* $M \otimes M^*(q_1, \ldots, q_r) \to \mathbb{C}$. ■

In the case $r = 2$ this theorem may be proved by the arguments of Section 3. The proof in the general case will be published elsewhere.

Now we show how to calculate the cohomology $H^*(\mathcal{L}(q_1, \ldots, q_r); M^*(q_1, \ldots, q_r))$. First of all we can decompose $M^*(q_1, \ldots, q_r)$ into irreducible representations with use of Theorem 5.6 Each of the irreducible components has the form $L_1 \otimes \ldots \otimes L_r$ where L_i is an irreducible quasi-finite representation of the Lie algebra Vir (q_i).

THEOREM 5.8. $H^*(\mathcal{L}(q_1, \ldots, q_r); L_1 \otimes \ldots \otimes L_r) \cong \otimes_r H^*(\mathcal{L}_1; L) \otimes \mathbb{C}[x_1, \ldots, x_{r-1}]$, *where* x_i *are generators of degree 2.*

The two factors in the right hand part of the last formula are obtained in the following way. The central composition series of the algebra $\mathcal{L}(q_1, \ldots, q_r)$ defines on it a topology, and the completion with respect to this topology is isomorphic to $\bar{\mathcal{L}}_1 \oplus \ldots \oplus \bar{\mathcal{L}}_1$ (r summands) $= \bar{\mathcal{L}}(q_1, \ldots, q_r)$, where $\bar{\mathcal{L}}_1$ is the completion of $\mathcal{L}_1$ with respect to topology defined by the central composition series. (The Lie algebra $\bar{\mathcal{L}}_1$ is nothing but the Lie algebra of formal vector fields in the line vanishing at the points 0 with the first derivative). The topological Lie algebra $\bar{\mathcal{L}}(q_1, \ldots, q_r)$ acts continuously in the space $L_1 \otimes \ldots \otimes L_r$ and hence there is a mapping θ: $H^*_c(\bar{\mathcal{L}}(q_1, \ldots, q_r); L_1 \otimes \ldots \otimes L_r) = {}_iH^*_c(\bar{\mathcal{L}}_1; Li) \to H^*(\mathcal{L}(q_1, \ldots, q_r); L_1 \otimes \ldots \otimes L_r)$. One can show that θ is a monomorphism. From the other hand, $H^*(\mathcal{L}(q_1, \ldots, q_r); L_1 \otimes \ldots \otimes L_r)$ is a module over the algebra $H^*(\mathcal{L}(q_1, \ldots, q_r); \mathbb{C})$. The last cohomology is known (see [11]). It is isomorphic to $\otimes^r H^*(\mathcal{L}_1) \otimes \mathbb{C}[x_1, \ldots, x_{r-1}]$. The generator x_i is represented by the cocycle

$$f\frac{d}{dz},\ g\frac{d}{gz} \to \int_{q_1}^{q_{i+1}} (f'g'' - f''g')\,ds$$

(where z is the coordinate in $\mathbb{C} = \mathbb{C}P^1 - \infty$). Thus $\mathbb{C}[x_1, \ldots, x_{r-1}]$ is a subalgebra in $H^*(\mathscr{L}(q_1, \ldots, q_r); \mathbb{C})$. The space $H^*(\mathscr{L}(q_1, \ldots, q_r); L_1 \otimes \ldots \otimes L_r)$ is a free $\mathbb{C}[x_1, \ldots, x_{r-1}]$-module, and the image of θ may be taken for a system of generators.

The family of Lie algebras $\mathscr{L}(q_1, \ldots, q_r)$ is a family of subalgebras in Vir(∞). Up to now we consider only the case when all the points $q_1, \ldots, q_r$ were distinct. When the point q_1 tends to q_2, the Lie algebra $\mathscr{L}(q_1, \ldots, q_r)$ tends to the Lie algebra $\mathscr{L}(q_1 \mid q_3, \ldots, q_r)$ consisting of vector fields which have four-fold zero at the point q_1 and two-fold zeroes in the point $q_3, \ldots, q_r$.

THEOREM 5.9. dim $H_i(\mathscr{L}(q_1, \ldots, q_r); M) =$ dim $H_i(\mathscr{L}(q_1 \mid q_3, \ldots, q_r); M)$ *for any $i \geqslant 0$ and any quasi-finite representation M of Vir(∞).* ∎

This will be proved elsewhere.

Now, if all the points $q_1, \ldots, q_r$ tend to one point then the Lie algebra $\mathscr{L}(q_1, \ldots, q_r)$ tend to a Lie algebra isomorphic to $\mathscr{L}_{2r-1}$.

CONJECTURE. dim $H_i(\mathscr{L}_{2r-1}; M) =$ dim $H_i(\mathscr{L}(q_1, \ldots, q_r); M)$. ∎

This is true for $M = \mathbb{C}$ (see [11]).

5. The most of the results of this section have analogs for affine Lie algebra. In particular, the algebra A_c is replaces with the algebra B_k generated by irreducible representations with integral dominant highest weights and with given central charge k. In this final subsection we consider in some details the case of the Lie algebra $\hat{\mathfrak{sl}}_2$. Under $\mathfrak{sl}\,2$ we understand here the central extension of the Lie algebra $\mathfrak{sl}_2 \otimes \mathbb{C}[t^{-1}, t]]$ (the symbol $\mathbb{C}[t^{-1}, t]]$ denotes Loran series in t, which are finite in the negative direction and possibly infinite in the positive direction). The central element of $\hat{\mathfrak{sl}}_2$ we denote by K, its action is representation we usually denote by k. In $\hat{\mathfrak{sl}}_2$ we consider the parabolic subalgebra $P = \mathbb{C}\cdot K \oplus \mathfrak{sl}_2 \otimes \mathbb{C}[[t]]$. Let π be a finite-dimensional irreducible representation of $\mathfrak{sl}_2$. Denote by $V_{\pi,k}$, where $k \in \mathbb{C}$, the representation of $\hat{\mathfrak{sl}}_2$ induced by the representation of P in the space π: K avts by multiplying with k, and $\mathfrak{sl}_2 \otimes \mathbb{C}[[t]]$ acts through the projection $\mathfrak{sl}_2 \otimes \mathbb{C}[[t]] \to \mathfrak{sl}_2$ ($t \to 0$). The representation $V_{\pi,k}$ is called a (generalized) Verma module. Irreducible representations $L_{\pi,k}$ of the Lie algebra $\hat{\mathfrak{sl}}_2$ with integral dominant

highest weight are parametrized by pairs (π, k), where k is a non-negative integer, and π is a finite-dimensional irreducible representation of $\mathfrak{sl}_2$ with the highest weight being less or equal to k. Clearly, $L_{\pi,k}$ is a quotient of $V_{\pi,k}$. There exists a resolution of $L_{\pi,k}$ composed of generalized Verma modules; for example, the resolution of $L_{\pi_0,k}$, where π_0 is the one-dimensional trivial representation, has the form: $0 \leftarrow L_{\pi_0,k} \leftarrow V_{\pi_0,k} \leftarrow V_{\pi_1,k} \leftarrow V_{\pi_2,k} \leftarrow \ldots$, where π_{2i} is the representation with the highest weight $2i(k+2)$, and π_{2i-1} is the representation with the highest weight $2i(k+2)-2$. The resolution of an arbitrary $L_{\pi,k}$ involves the modules $V_{s_i(\pi,k),k}$ where $s_i(\pi,k)$ is the representation of $_2$ with the highest weight $2i(k+2)-1+(-1)^i(1+\chi)$, where χ is the highest weight of π.

The Lie algebra $\hat{\mathfrak{sl}}_2$ contains a subalgebra $\mathfrak{sl}_2 \otimes t \cdot \mathbb{C}[[t]]$. The cohomology $H^*(\mathfrak{sl}_2 \otimes t \cdot \mathbb{C}[[t]]; L_{\pi,k})$ is an $\mathfrak{sl}_2$-module, for $\mathfrak{sl}_2$ is the quotient of the normalizer of $\mathfrak{sl}_2 \otimes t \cdot \mathbb{C}[[t]]$ in $\hat{\mathfrak{sl}}_2$ over $\mathfrak{sl}_2 \otimes t \cdot \mathbb{C}[[t]]$.

PROPOSITION. *$H^i(\mathfrak{sl}_2 \otimes t \cdot \mathbb{C}[[t]]; L_{\pi,k})$ is an irreducible $\mathfrak{sl}_2$-module isomorphic to $s_i(\pi, k)$.*

Proof: consider the resolution of $L_{\pi,k}$ constructed above.

Let $p_1, \ldots, p_n$ be n distinct points of $\mathbb{C}P^1$, $U = \mathbb{C}P^1 - \{p_1, \ldots, p_n\}$ $\mathfrak{sl}_2(U)$ be the Lie algebra of rational currents $\mathbb{C}P^1 \to \mathfrak{sl}_2$ without poles in U. Consider the completion of the Lie algebra $\mathfrak{sl}_2(U)$with respect to the adic topology defined by the set $p_1, \ldots, p_n$. The completion $\overline{\mathfrak{sl}}_2(U)$ is isomorphic to the direct sum of n copies of the Lie algebra $\mathfrak{sl}_2 \otimes \mathbb{C}[t^{-1}, t]]$. Each of the points p_i defined a 2-cocycle of the Lie algebra $\overline{\mathfrak{sl}}_2(U)$, the sum of these cocycles defines the central extension of $\overline{\mathfrak{sl}}_2(U)$ which is denoted by $\hat{\mathfrak{sl}}_2(p_1, \ldots, p_n)$. The Lie algebra $\mathfrak{sl}_2(U)$ is embedded into $\hat{\mathfrak{sl}}_2(p_1, \ldots, p_n)$.

Fix an integer k. Let $L_{\nu_1,k}, \ldots, L_{\nu_n,k}$ be a set of irreducible representations of the Lie algebra $\hat{\mathfrak{sl}}_2$. The tensor product $L = L_{\nu_1,k} \otimes L \otimes \ldots \otimes L_{\nu_n,k}$ is a module over the Lie algebra $\hat{\mathfrak{sl}}_2(p_1, \ldots, p_n)$ and hence over the Lie algebra $\mathfrak{sl}_2(U)$. ■

The proofs of the following results will be published elsewhere.

THEOREM 5.10. *The space $H_0(\mathfrak{sl}_2(U); L)$ is finite-dimensional, and its dimension is equal to* $\Sigma(-1)^i \dim \operatorname{Hom}_{\mathfrak{sl}_2}(\pi_i, \nu_1 \otimes \nu_2 \otimes \ldots \otimes \nu_n)$; here π_i are the representations involved in the resolution of $L_{\pi_0,k}$ – see above. The space $H^*(\mathfrak{sl}_2(U); L^*)$ is isomorphic to $H^0(\mathfrak{sl}_2(U); L^*) \otimes H^*(\mathfrak{sl}_2(U); \mathbb{C})$. ■

Notice that $H^*(\mathfrak{sl}_2(U); \mathbb{C}) \cong H^*(\mathfrak{sl}_2; \mathbb{C}) \otimes \mathbb{C}[y_1, \ldots, y_{n-1}]$, where

y_i are generators of degree 2. The similar statement is valid for currents on any complex curve. (See [3]).

Let $q_1, \ldots, q_n$ be a set of distinct and different from ∞ points of $\mathbb{C}P^1$, $U = P^1 - \infty$ and $\mathfrak{sl}_2(U, q_1, \ldots, q_n)$ be the subalgebra of $\mathfrak{sl}_2(U) = \mathfrak{sl}_2 \otimes \mathbb{C}[t]$ consisting of currents vanishing at the points $q_1, \ldots, q_n$. The Lie algebra $\mathfrak{sl}_2$ $(U, q_1, \ldots, q_n)$ is embedded into $\mathfrak{sl}_2^{\wedge}(\infty)$. Therefore the $\mathfrak{sl}_2^{\wedge}$-module $L_{\pi,k}$ is an $\mathfrak{sl}_2(U, q_1, \ldots, q_n)$-module as well. Similarly $\mathfrak{sl}_2(U, q_1, \ldots, q_n)$ is embedded into $\mathfrak{sl}_2^{\wedge}(q_1, \ldots, q_n)$.

THEOREM 5.11. *The space* $H_i(\mathfrak{sl}_2(U, q_1, \ldots, q_n); L_{\pi,k})$ *is dual to the space*

$$\bigoplus_{\pi_1 \ldots \pi_n} [H^i(\mathfrak{sl}_2(U, q_1 \ldots q_n); L_{\pi_1,k} \otimes \ldots \otimes L_{\pi_n,k}) \otimes$$

$$\otimes H^0(\mathfrak{sl}_2(\mathbb{C}P^1 - \infty, q_1, \ldots, q_n); L_{\pi,k} \otimes L_{\pi_1,k} \otimes \ldots \otimes L_{\pi_n,k})$$

■

In this statement $L_{\pi,k}$ is supposed to be situated at ∞ and $L_{\pi_i,k}$ is supposed to be situated at the point p_i.

The space $H^*(\mathfrak{sl}_2(U, q_1, \ldots, q_n); L_{\pi_1,k} \otimes \ldots \otimes L_{\pi_n,k})$ is calculated with use of the following theorem.

THEOREM 5.12. $H^*(\ _2(U, q_1, \ldots, q_n); L_{\pi_1,k} \otimes \ldots \otimes L_{\pi_n,k}) \cong$

$$\cong \otimes^i H^*(\mathfrak{sl}_2 \otimes t \cdot \mathbb{C}[t]; L_{\pi_i,k}) \otimes \mathbb{C}[x_1, \ldots, x_{n-1}],$$

where

$$\deg x_i = 2.$$

■

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